Tâtonnement Dynamics for Fisher Markets with Chores

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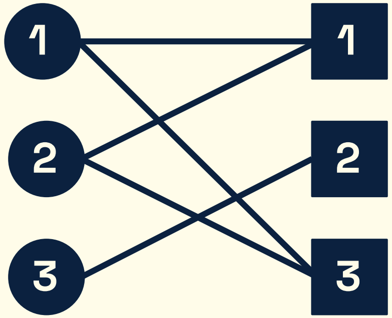
Chores Fisher Markets

Agents
↪ take on chores to earn money
↪ incur disutilities for a bundle of chores assigned to her
↪ has an earning requirement



Chores
↪ unit supply
↪ its price is the payment-per-unit of the chore done

Competitive Equilibrium

A price and an allocation (p, x) are at CE iff they satisfy the following conditions:

- **Optimal bundles:** Each agent gets a disutility-minimizing chore bundle subject to her earning requirement
- **Market clearing:** Every chore is completely allocated

CCH and Convex CES Disutilities

- We consider general continuous, convex, 1-homogeneous (CCH) disutility function that are strictly increasing in each coordinate (in many results this assumption can be replaced by weaker assumptions such as local non-satiation)

- The main class of disutility functions studied is convex CES disutilities:

$$d_i(\mathbf{x}_i) = \left(\sum_{j=1}^{m} d_{ij} x_{ij}^{\rho}\right)^{\frac{1}{\rho}}$$

Divergence of (Naïve) Tâtonnement

- A direct analogous tâtonnement dynamics corresponding to the goods case, called *naïve tâtonnement*:

$$\begin{aligned} \text{Pick } \mathrm{z}^t &\in Z(\mathrm{p}^t) \\ \mathrm{p}^{t+1} &= \mathrm{p}^t - \eta^t \mathrm{z}^t \end{aligned} \qquad \text{(Naïve Tâtonnement)}$$

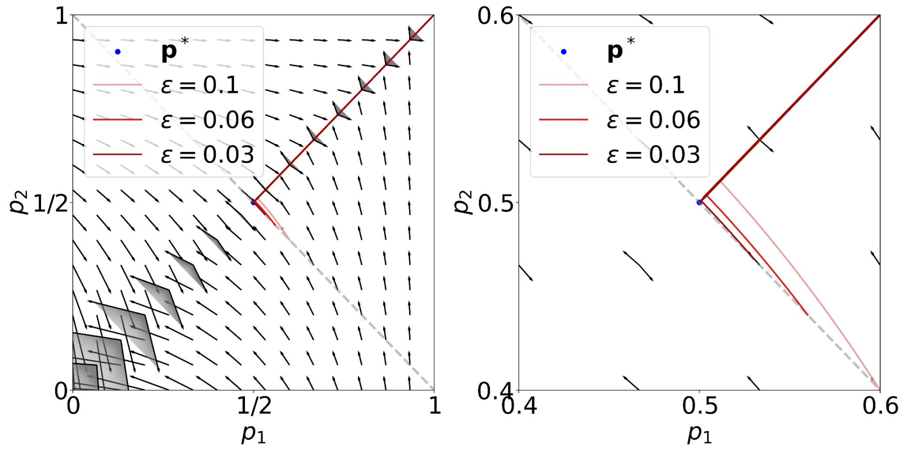
- Even though the analogous tâtonnement converges for a wide range of utility functions, Naïve Tâtonnement does not work in chores case

Divergence of (Naïve) Tâtonnement

- Observation: *the excess demand has conflict effects*

Goods	Chores
Price increases → less attractive → demand decrease	Price increases → more attractive → demand increase
Price increases → less demand given the same amount of money	Price increases → less demand given the same amount of money
Demand < supply: price is high	Demand < supply: price is either high or low

Divergence of (Naïve) Tâtonnement



Divergence of Naïve Tâtonnement

For any $\rho \in [1, \infty)$, there is a ρ-CES chores Fisher market with a unique CE price vector p^*, such that for every initial price vector $p^0 \in \{ p \geq 0 \mid \sum_{j \in [m]} p_j \geq \sum_{i \in [n]} B_i, p \neq p^* \}$, the (discrete-time) naïve tâtonnement dynamics equipped with ***any*** sequence of positive stepsizes $\{ \eta^t \}_{t \geq 0}$ and ***any*** tie-breaking rule diverges from p^*.

Relative Tâtonnement and its Convergence

- We propose an alternative tâtonnement dynamics with a mild and interpretable modification, called *relative tâtonnement*:

$$\begin{aligned} &\text{Pick } \mathbf{z}^t \in Z(\mathbf{p}^t) \\ \tilde{\mathbf{z}}^t &= \mathbf{z}^t - \frac{1}{m}\mathbf{1}_m^\top \mathbf{z}^t \\ \mathbf{p}^{t+1} &= \mathbf{p}^t - \eta^t \tilde{\mathbf{z}}^t \end{aligned} \qquad \text{(Relative Tâtonnement)}$$

- Our price adjustment mechanism responds to the relative excess demands *compared to the averaging level*, and thus maintain the total prices at a constant level

$$H_B := \left\{ \mathbf{p} \in \mathbb{R}^m \,\middle|\, \sum_{j=1}^{m} p_j = \sum_{i=1}^{n} B_i \right\}$$

Relative Tâtonnement and its Convergence

- We utilize the function showing up in the chores EG dual program (generalized to the general CCH disutilities)

$$f(\mathbf{p}) = -\sum_{j=1}^{m} p_j + \sum_{i=1}^{n} B_i \log\left(\max_{\mathbf{x}_i \geq 0 : d_i(\mathbf{x}_i) \leq 1} \langle \mathbf{p}, \mathbf{x}_i \rangle \right)$$

- We consider the restriction: $f|_{H_B}(\mathbf{p}) : \Delta_B \to \mathbb{R}$ as a potential function

- Then, relative tâtonnement is connected to a subgradient dynamical system associated with the unconstrained optimization problem. However, this potential function is *nonconvex* and *nonsmooth*, in which the chain rule can fail

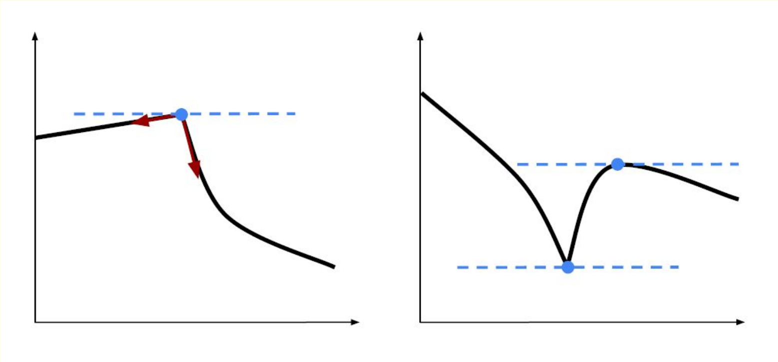
$$\min_{\mathbf{p} \in \mathbb{R}^m} f|_{H_B}(\mathbf{p})$$

Relative Tâtonnement and its Convergence

- To prove the convergence (or even compute the "gradient") we need to explore the properties of the potential function

$$f(\mathbf{p}) = -\sum_{j=1}^{m} p_j + \sum_{i=1}^{n} B_i \log\left(\max_{\mathbf{x}_i \geq 0 : d_i(\mathbf{x}_i) \leq 1} \langle \mathbf{p}, \mathbf{x}_i \rangle\right)$$

- Even though the function is nonconvex and nonsmooth, it is *subdifferentially regular*—it is important in proving the equivalence and convergence to chores CE



Discrete-time Relative Tâtonnement

- Discrete-time variant can be seen as an approximation of the continuous-time counterpart

Convergence of Discrete-time Relative Tâtonnement

In a chores Fisher market with CCH disutilities that are strictly increasing in each coordinate, let $\{p^k\}, k = 0, 1, \ldots$ be a sequence of iterates generated by (Relative Tâtonnement) with proper stepsizes and any initial point $p^0 \in \Delta_B$. Then, every limit point of the iterates $\{p^k\}, k = 0, 1, \ldots$ is a stationary point of the potential function and thus a CE.

EG Dual Program for general CCH disutilities

- The above proofs also lead to an optimization program for general CCH disutilities

$$\begin{aligned} \min_{\mathbf{p}\geq 0} \quad & -\sum_{j=1}^{m} p_j + \sum_{i=1}^{n} B_i \log\left(\max_{\mathbf{x}_i \geq 0 : d_i(\mathbf{x}_i) \leq 1} \langle \mathbf{p}, \mathbf{x_i} \rangle\right) \\ \text{s.t.} \quad & \sum_{j=1}^{m} p_j = \sum_{i=1}^{n} B_i. \end{aligned} \qquad \text{(general EG Dual)}$$

General EG Dual Program

There is a one-to-one mapping between the set of equilibrium prices in the chores Fisher market and the set of primal points to (general EG Dual) that satisfies the KKT conditions. Moreover, (general EG Dual) has a finite lower bound.

$\tilde{O}(1/\varepsilon^2)$ Convergence to Approximate CE

- The main challenge is the dramatic demand changes that can be triggered by a very small change of price

- As such, the signal of the relative excess demand can be very large, even very close to a stationary point, making the dynamics hard to detect how close a stationary point is

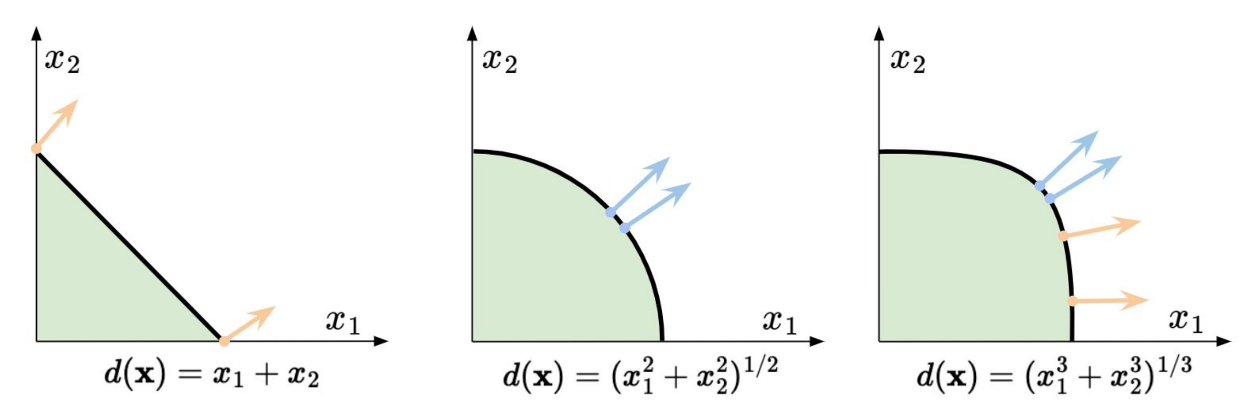
- We need to establish the *smoothness* of the potential function. We focus on the inner-log term:

$$\max_{\mathbf{x}_i \geq 0 : d_i(\mathbf{x}_i) \leq 1} \langle \mathbf{p}, \mathbf{x}_i \rangle$$

  which is the *gauge dual* of the primal disutility function d_i

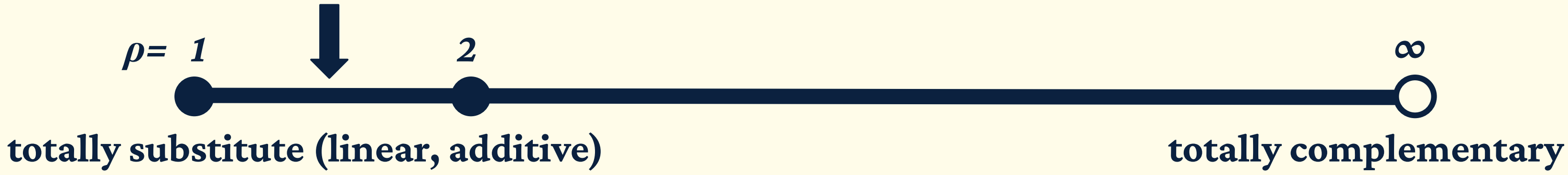
$\tilde{O}(1/\varepsilon^2)$ Convergence to Approximate CE

- The smoothness of the gauge dual can be understood geometrically via the curvature of the boundary of the unit disutility ball $\{\mathbf{x}_i : d_i(\mathbf{x}_i) \leq 1\}$



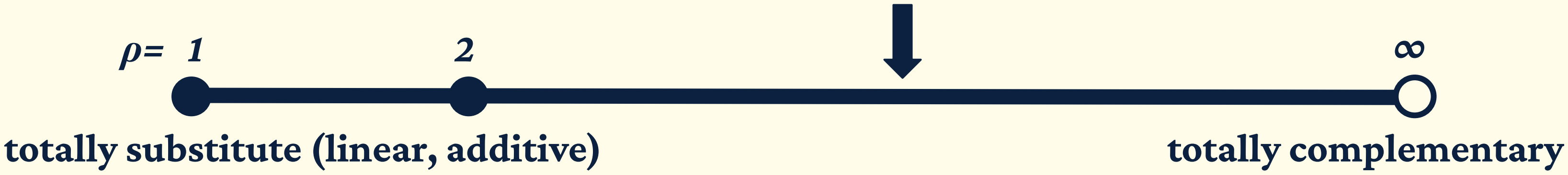
$\tilde{O}(1/\varepsilon^2)$ Convergence to Approximate CE

- Convex CES disutility functions are not strictly/strongly convex
- When $\rho \in (1, 2]$, the square of the disutility function is strongly convex, and the corresponding unit sublevel set coincides with that of the original disutility
- We establish such a smoothness globally for a broader class of CCH disutilities, including many other interesting disutility functions, e.g., $d_i(\mathbf{x}_i) = \|A\mathbf{x}_i\|_2$ where A is a full rank matrix.



$\tilde{O}(1/\varepsilon^2)$ Convergence to Approximate CE

- When $\rho \in (2, \infty)$, the level sets become flat along coordinate axes
- *locally smooth region*: all prices are bounded away from zero
- With proper chosen step-size:
  - Phase 1: tâtonnement iterates enter the locally smooth region within polynomially many iterations
  - Phase 2: tâtonnement iterates stay in the region by a discrete-time thresholding argument



Stability of Individual CE

- Informally, an equilibrium price vector p^* is said to be *locally stable* with respect to a continuous-time dynamical system if there exists a non-empty feasible neighborhood around p^* such that every trajectory starting in the neighborhood converges to p^*

- We provide several characterizations of *locally stable CE* when agents have linear disutility functions: e.g., combinatorial structure: a CE is locally stable if and only if the minimum pain-per-buck (MPB) graph (with non-zero allocation) is *connected*

Stability of Individual CE

- An interesting characterization is that the CE that maximizes Nash welfare, i.e., the product of individual disutilities, is always locally stable:

  When agents have linear disutility functions, the Nash welfare-maximizing competitive equilibria are locally stable.

- Stability criterion demonstrates that a Nash welfare-maximizing CE is always desirable from a stability perspective

Thank you for your listening!

Any questions?