Convergence of Tâtonnement for Linear Fisher Markets

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Fisher Market

- **Fair division**
- **Online resource allocations**

Linear Fisher Market

Each buyer i:

- has a fixed budget $B_i > 0$
- has a linear utility function $u_i(x_i) = v_{i1}x_{i1} + \cdots + v_{im}x_{im}$

Each item j:

- is divisible
- has a unit supply: $s_j = 1$
- has a price $p_j \geq 0$

Market Equilibrium (ME)

Market equilibrium is a pair of

(equilibrium price $p^\star$, equilibrium allocation $x^\star$)

satisfying

- **[Buyer optimality]**
  Each buyer i is allocated her optimal bundle of items
  ($x_i^\star$ maximizes buyer i's utility under her budget constraint)

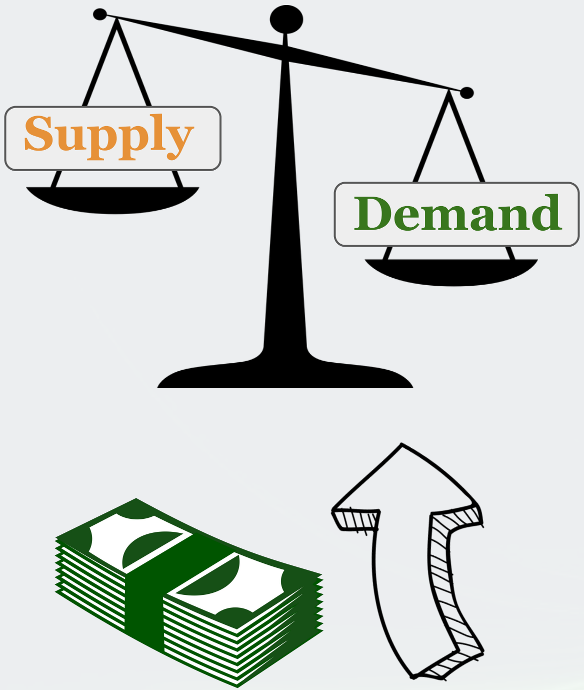
- **[Market clearance]**
  Each item j is fully allocated (the total demand $\sum_{i=1}^{n} x_{ij}^\star$ = the total supply 1)

Our Focus: Economic Dynamics

- **Economic dynamics?**
- **Converge to ME?**
- **How fast is the convergence?**

Tâtonnement

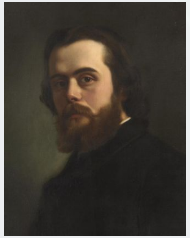
A **simple**, **natural**, and **decentralized** price-adjustment process where prices of items are adjusted to reflect buyers' excess demands

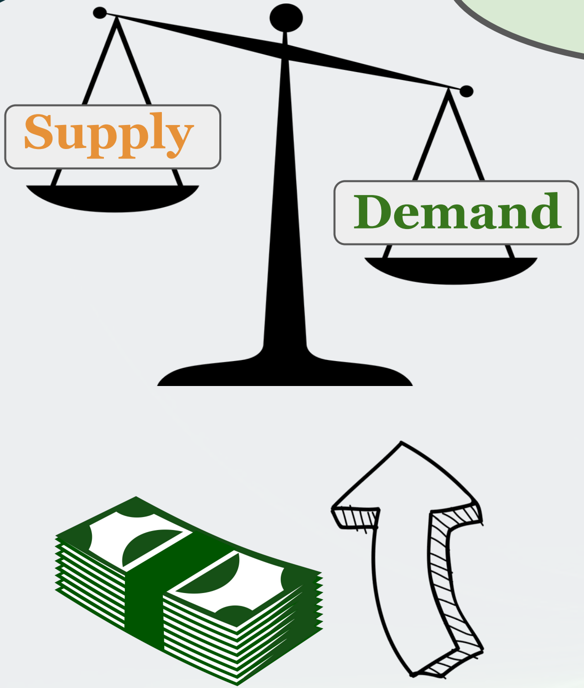


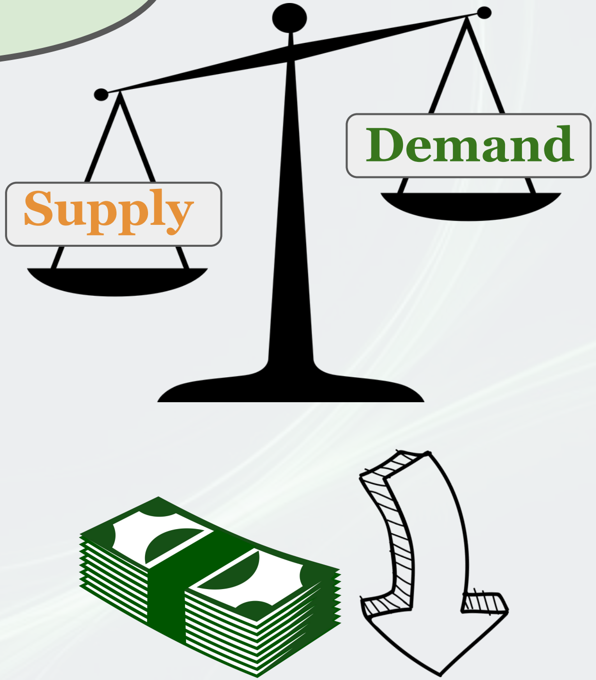
Tâtonnement

1874

“Groping” in French







Our Goal

Does **Tâtonnement** converge to an (approximate) **Market Equilibrium** in **Linear Fisher Markets** ?

Previous Results

Tâtonnement	Papers	Market Type	Result
Additive*	Codenotti, McCune, and Varadarajan (2005)	Exchange market WGS, unique demand	Poly-time to approx. ME
Multiplicative	Avigdor-Elgrabli, Rabani, and Yadgar (2014)	Nested CES-Leontief $\rho \in (-\infty, 0) \cup (0, 1)$	$\mathcal{O}\left(\frac{1}{\epsilon^3}\right)$
Multiplicative*	Cole and Fleischer (2008)	Exchange market, WGS	$\mathcal{O}\left(E \log \frac{1}{\epsilon}\right)$
	Cheung, Cole, and Rastogi (2012)	CES ($-1 < \rho \leq 0$) Nested non-lin. CES	$\mathcal{O}\left(\log \frac{1}{\epsilon}\right)$
Entropic*	Cheung, Cole, and Devanur (2019)	CES ($-\infty \leq \rho < 0$)	$\mathcal{O}\left(\log\left(1/\epsilon\right)\right)$ $\mathcal{O}\left(1/\epsilon\right)$ if $\rho = -\infty$
	Goktas, Zhao, and Greenwald (2023)	CCNH	$\mathcal{O}\left(\frac{1+E^2}{\epsilon}\right)$
Entropic*	Cole and Tao (2019)	Linear, Large market assum.	Lin conv to approx. ME
Additive	This work	Linear	Lin conv to approx. ME

Our Results

- **Approximate ME tends to exact as the stepsize decreases;**
- **Tâtonnement converges to the approximate ME in a linear rate;**

For any $\boldsymbol{\varepsilon}$-approximation market equilibrium,
we can find a stepsize $\boldsymbol{\eta}>0$ such that tâtonnement converges to that approximation level

Convex Potential Function

$$\min_{p \geq 0} \ \varphi(p) := \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} B_i \ln\left(\min_{j'} \frac{p_{j'}}{v_{ij'}}\right)$$

Equilibrium price $p^\star$ ⟺ Optimal solution to $\min_{p \geq 0} \varphi(p)$

Tâtonnement ⟺ Subgradient descent on $\min_{p \geq 0} \varphi(p)$

Proof Sketch

Step 1: Prices are lower bounded

- Ensures the excess demands are bounded
- Previously artificial lower bounds were often imposed
- We showed there is a positive lower bound once a small enough stepsize is chosen

Proof Sketch

Step 2: Quadratic Growth (QG) Function

- Construct a smooth function that lower bounds the convex (nonsmooth) function $\varphi(p)$, and has the same optimum point as $\varphi(p)$
- Show the auxiliary function has QG property
- Show $\varphi(p)$ also has QG property

Proof Sketch

Step 3: Linear Convergence Rate

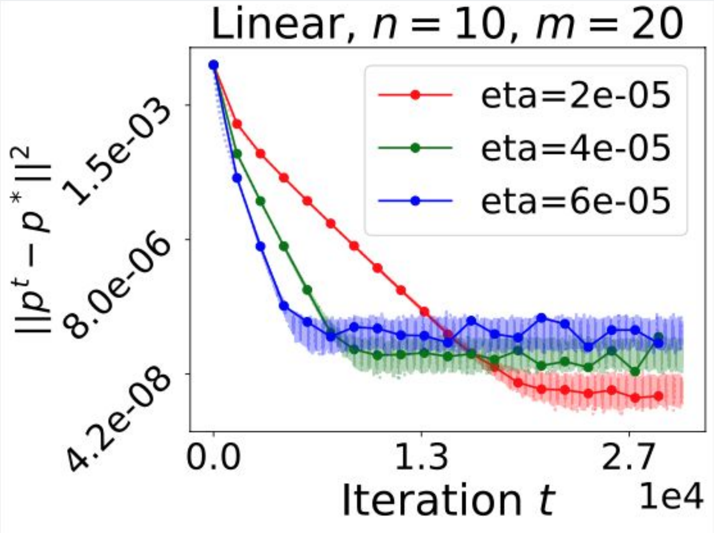
- Follows classic proof under bounded subgradient and QG property
- Explicitly shows a convergence rate and the radius of the final approximation level in terms of the stepsizes

Our Results

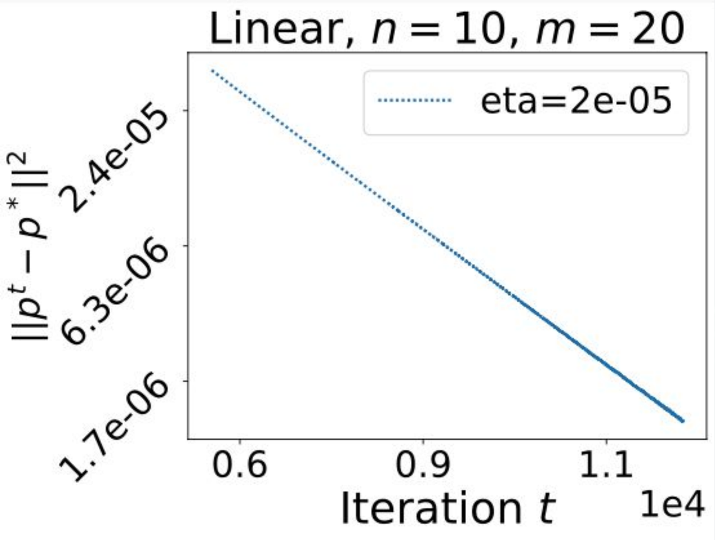
Tâtonnement converges to
an **approximate market equilibrium**
(that tends to be exact as the stepsize decreases)
at **a linear rate** in the **linear Fisher market**
once the stepsize is not large.

Analogous results hold for quasi-linear Fisher market
which connects to the applications in large-scale Internet markets

Our Results

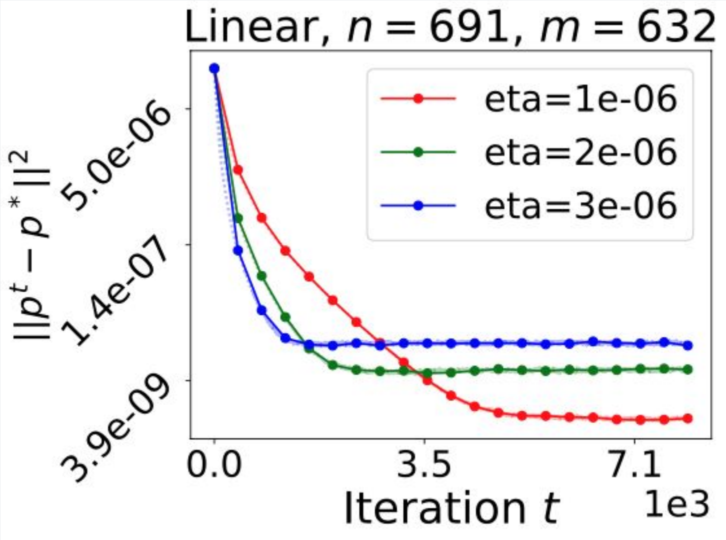


Linear, $n = 10$, $m = 20$
eta=2e-05
eta=4e-05
eta=6e-05
$||p^t - p^*||^2$
1.5e-03
8.0e-06
4.2e-08
0.0
1.3
2.7
Iteration t
1e4

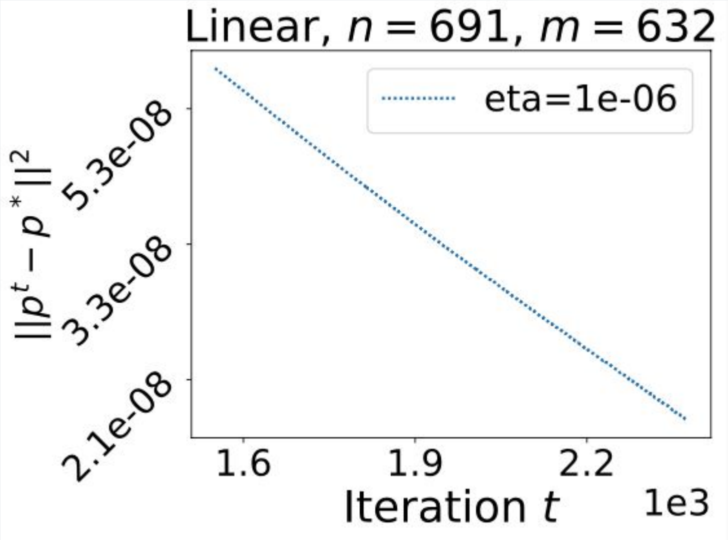


Linear, $n = 10$, $m = 20$
eta=2e-05
$||p^t - p^*||^2$
2.4e-05
6.3e-06
1.7e-06
0.6
0.9
1.1
Iteration t
1e4

Our Results



Linear, $n = 691$, $m = 632$
eta=1e-06
eta=2e-06
eta=3e-06
$||p^t - p^*||^2$
5.0e-06
1.4e-07
3.9e-09
0.0
3.5
7.1
Iteration t
1e3



Linear, $n = 691$, $m = 632$
eta=1e-06
$||p^t - p^*||^2$
5.3e-08
3.3e-08
2.1e-08
1.6
1.9
2.2
Iteration t
1e3

Thank you for listening!!!