Competitive Equilibrium for Chores: from Dual Eisenberg-Gale to a Fast, Greedy, LP-based Algorithm

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Competitive Markets for Goods vs. Chores

Goods **Chores**

Chores Fisher markets

- n agents
- m divisible chores
- B_i: amount agent i **needs to earn**
- 1 divisible unit of supply for each chore **(must be divided)**
- d_{ij}: disutility of 1 unit of chore $j \rightarrow$ agent i
- **(Linear disutility function)** Agent i's disutility is $\sum_{j=1}^{m} d_{ij} x_{ij}$

Competitive Equilibrium (CE)

- CE = **Price** p + **Allocation** x which satisfies
  - Agent optimality: every agent gets her utility-maximizing (disutility-minimizing) bundle among her “affordable” bundles
  - Market clearance: demand = supply
- In goods case:
  - CE are captured by the well-known Eisenberg-Gale (EG) convex program

$$\begin{aligned} \max_{x \geq 0} \quad & \sum\nolimits_{i \in [n]} B_i \log \left(\sum\nolimits_{j \in [m]} v_{ij} x_{ij} \right) \\ \text{s.t.} \quad & \sum\nolimits_{i \in [n]} x_{ij} = 1 \qquad \forall j \in [m] \end{aligned} \qquad \text{(EG Program)}$$

  - Many algorithms have been established based on the EG program
  - There exist polynomial time algorithms to compute a CE

Competitive Equilibrium (CE) for Chores

- CE with chores is significantly different and harder:
  - The set of CE can be non-convex and disconnected
  - Computing CE in exchange markets with chores is PPAD-hard
  - Polynomial-time algorithms are only known for computing approximate CE
- Bogomolnaia et al. [2017] showed that every **stationary point** of

$$\begin{aligned} \inf_{x \geq 0} \quad & \sum_{i \in [n]} B_i \log \left(\sum_{j \in [m]} d_{ij} x_{ij} \right) \\ \text{s.t.} \quad & \sum_{i \in [n]} x_{ij} = 1 \qquad \forall j \in [m] \qquad [\mathbf{p_j}] \end{aligned} \qquad \text{(Chores Primal)}$$

corresponds to a CE.

Difficulty: Unbounded-objective Directions

$$\begin{aligned} \inf_{x \geq 0} \quad & \sum_{i \in [n]} B_i \log \left(\sum_{j \in [m]} d_{ij} x_{ij} \right) & & \text{(Chores Primal)} \\ \text{s.t.} \quad & \sum_{i \in [n]} x_{ij} = 1 \qquad \forall j \in [m] \qquad [\mathbf{p_j}] & & \end{aligned}$$

- There are **unbounded-objective directions**: if $\sum_{j \in [m]} d_{ij} x_{ij} \to 0$ for some agent i, then the objective $\to -\infty$
- Standard iterative optimization methods (e.g. proximal gradient descent) move along these unbounded-objective directions.

Our Goal

- Existing methods involve solving **non-linear convex programs** at every step
- Goal:
  - Simple, practically-implementable approach (do not rely on solving non-linear programs)
  - Better theoretical guarantees.
- Our contributions:
  - A new program that captures all CE as its KKT points, **without poles / any unbounded-objective direction**
  - A fast algorithm to find the desired KKT points of the new program

"Dual" of Chores Primal

- The Lagrangian dual of EG in the goods case:

$$\begin{aligned} \min_{p,\beta\geq 0} \quad & \sum_j p_j - \sum_i B_i \log(\beta_i) \\ \text{s.t.} \quad & p_j \geq v_{ij}\beta_i \quad \forall\, i \in [n], j \in [m]. \qquad [\mathbf{x_{ij}}] \end{aligned}$$

- We "give" the "dual" of the Chores EG:

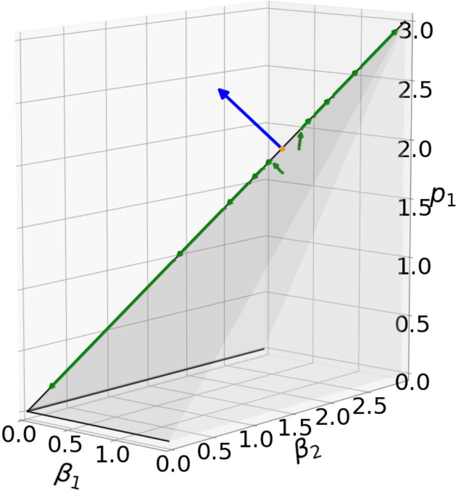
$$\begin{aligned} \sup_{p,\beta\geq 0} \quad & \sum_j p_j - \sum_i B_i \log(\beta_i) \\ \text{s.t.} \quad & p_j \leq d_{ij}\beta_i \quad \forall\, i \in [n], j \in [m]. \qquad [\mathbf{x_{ij}}] \end{aligned} \qquad \text{(Chores Dual)}$$

“Dual” of Chores Primal

- **Theorem**: If (x, p) is a KKT point of Chores Primal, then (p, β, x) is a KKT point of Chores Dual where $\beta_i = B_i / \sum_j d_{ij} x_{ij}$, and vice versa. Furthermore, the primal and dual program objectives are equal at every KKT point.

- Still, there are unbounded-objective directions in Chores Dual.

"Dual" of Chores Primal: Unbounded-objective Directions



$$\begin{pmatrix} p \le 2\beta_1 \\ p \le \beta_2 \\ \beta_1, \beta_2, p \ge 0 \end{pmatrix}$$

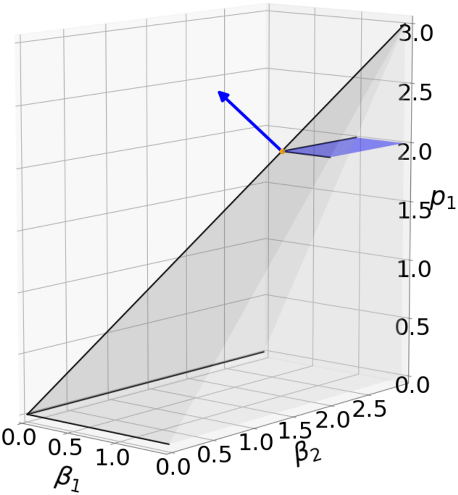
A Crucial "Redundant" Constraint

- We (surprisingly) fix this issue by adding an additional constraint to Chores Dual:

$$\begin{aligned} \max_{p,\beta\geq 0} \quad & \sum_j p_j - \sum_i B_i \log(\beta_i) \\ \text{s.t.} \quad & p_j \leq d_{ij}\beta_i \quad \forall\, i,j \qquad\qquad \text{(Chores Dual Redundant)} \\ & \sum_j p_j = \sum_i B_i. \end{aligned}$$

- **Theorem**: There is a one-to-one correspondence between CE of the chores Fisher market and the KKT points of Chores Dual Redundant.

Polyhedron with the redundant constraint on an instance



$$\begin{pmatrix} p \leq 2\beta_1 \\ p \leq \beta_2 \\ p = 2 \\ \beta_1, \beta_2, p \geq 0 \end{pmatrix}$$

Algorithm: Greedy Frank Wolfe (GFW)

- Next, we want a simple, efficient, fast algorithm to find **a KKT point** of Chores Dual Redundant.
- This is a convex maximization problem over a polyhedron:

$$\max_{x \in X} f(x), \qquad \text{(Convex Max over a Polyhedron)}$$

where f is convex and X is a polyhedron.

- A special instance of the Frank-Wolfe algorithm was previously proposed to solve such problems.

Recall: Frank Wolfe (FW) Algorithm

- For general $\max_{x\in\mathcal{X}} f(x)$ problem, FW algorithm computes

$$\begin{aligned} v^i &= \arg\max_{x\in\mathcal{X}} f(x^i) + \langle \nabla f(x^i), x - x^i \rangle \\ &= \arg\max_{x\in\mathcal{X}} \langle \nabla f(x^i), x \rangle . \end{aligned} \tag{LMO}$$

  and then set

$$x^{i+1} = (1-\gamma^i)x^i + \gamma^i v^i.$$

- With proper stepsizes, the limit point of $\{x^i\}$ is a stationary point
- Attractive when LMO is easy to compute (e.g. $\mathcal{X}$ is a polyhedron)

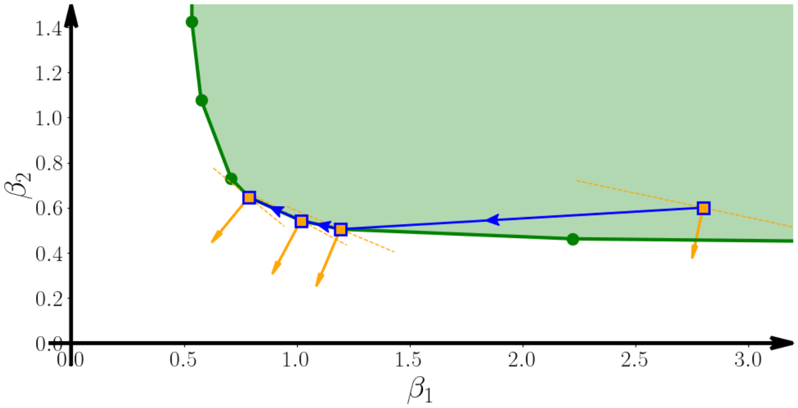
Algorithm: Greedy Frank Wolfe (GFW)

- Greedy Frank Wolfe for Convex Max over a Polyhedron:

$$\gamma^i = 1 \quad \text{for all } i.$$

- **Theorem** [Mangasarian, 1996]: If f is bounded above, GFW generates a sequence of iterates with strictly increasing objective value, and thus finds a stationary point in a finite number of iterations.

Algorithm: Greedy Frank Wolfe



Theoretical Results: Approximate Equilibrium

Theorem

The GFW finds an ε-approximate CE in $\mathcal{O}\left(n\frac{\max_i B_i}{\min_i B_i}\log(mD)\frac{1}{\varepsilon^2}\right)$ iterations. If all agents' budgets are 1, *the time complexity is* $\mathcal{O}\left(n\log(mD)\frac{1}{\varepsilon^2}\right)$.

- Comparison:

GFW	Exterior point method	Combinatorial algorithm
$\tilde{\mathcal{O}}\left(\frac{n}{\varepsilon^2}\right)$	$\tilde{\mathcal{O}}\left(\frac{n^3}{\varepsilon^2}\right)$	$\tilde{\mathcal{O}}\left(\frac{nm}{\varepsilon^2}\right)$

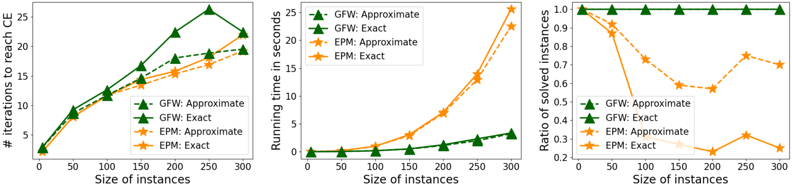
We showed a general result on general log-type convex maximization problem:

$$\max_{x\in\mathbb{R}^n_{>0}} \quad g(x) := -\sum_{i=1}^{n} a_i \log x_i \qquad \text{(log-type convex maximization)}$$
$$\text{s.t.} \quad x \in \mathcal{X}.$$

Numerical Experiments

Simulated data:

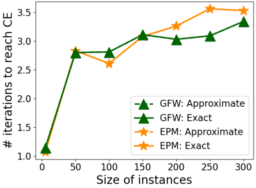
- Distribution of disutilities: uniform in $0 - 1$ and others
- Size of instances n: n agents $\times$ n chores
- All budgets are 1

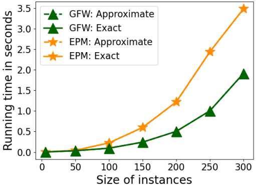


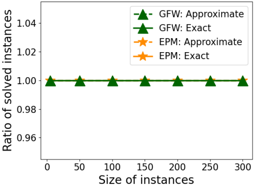
Numerical Experiments

Real data:

- AAMAS 2021 bidding data
- Papers: chores; Reviewers: agents
- Set disutilities: yes (1) $<$ maybe (3) $<$ no response (5) $<$ no (7) $<$ conflict (4000)







Simplex-like variant of the GFW algorithm.

- At every iteration of GFW, we solve an LP to optimality, which can be done with the simplex algorithm;
- We could instead update the GFW objective at every simplex iteration (pivot).

(#Agents, #Chores)	#Pivots in GFW	#Pivots in "Simplex"
(20, 20)	70.4	21.6
(60, 60)	334.5	157.9
(100, 100)	3492.6	327.6

Table: The number of simplex pivots required to find an exact CE. We compute averages over 10 random instances. The disutilities are sampled from the uniform $[0, 1]$ distribution.

Thanks!

Thanks for listening!

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