On the Convergence of Tâtonnement for Linear Fisher Markets

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Takeaways

1. For a sufficiently small step size, the prices given by the tâtonnement process are guaranteed to converge linearly to equilibrium prices, up to a small approximation radius that depends on the stepsize.
2. Similar convergence results hold for tâtonnement in quasi-linear Fisher markets.
3. Numerical demonstrations.

Preliminaries

Linear Fisher markets

- ☐ n buyers and m divisible items;
- ☐ Budgets $B_i > 0, \forall\, i$ ($B = \sum_i B_i, B_{\min} = \min_i B_i$);
- ☐ Valuations $v_{ij} \geq 0, \forall\, i, j$;
- ☐ An allocation x_i gives buyer i a utility of $u_i(x_i) = \langle v_i, x_i \rangle$;
- ☐ A unit supply of each item j.

Competitive / Walrasian / Market Equalibrium (ME)

A Fisher market reaches an ME if allocations $x_i^*, \forall\, i$ and prices p^* satisfy

- Buyer optimality: $x_i^* \in \operatorname{Argmax}_{x_i : \langle p, x_i \rangle \leq B_i,\, x_i \geq 0} \langle v_i, x_i \rangle, \forall\, i \in [n]$,
- Market clearance: $\sum_{i=1}^n x_{ij}^* = 1, \forall\, j \in [m]$.

Tâtonnement

- ☐ A simple, natural, and decentralized price-adjustment process
- ☐ The price increases if the demand exceeds the supply, and decreases otherwise
- ☐ Starting with $p^0 \geq 0$, update via
  - ▷ Pick $x_i^t \in \operatorname{Argmax}_{x_i : \langle p, x_i \rangle \leq B_i,\, x_i \geq 0} \langle v_i, x_i \rangle \quad \forall\, i \in [n]$
  - ▷ $z_j^t = \sum_{i=1}^n x_i^t - 1 \quad \forall\, j \in [m]$
  - ▷ $p_j^{t+1} = p_j^t + \eta z_j^t \quad \forall\, j \in [m]$

Competitive equilibrium and Tâtonnement have been studied since 1874 (exactly 150 years ago)

Subgradient Descent Method

- ☐ A method to solve non-smooth convex minimization problem ($\min_{x \in \mathcal{X}} \varphi(x)$)
- ☐ Starting with $x^0 \in \mathcal{X}$, update via
  - ▷ Pick $g(x^t) \in \partial \varphi(x^t)$
  - ▷ $x^{t+1} = \Pi_{\mathcal{X}}\left(x^t - \eta g(x^t)\right)$

Quadratic Growth

A closed convex function φ has α-quadratic growth property if

$$\varphi(x) - \varphi(\Pi_{\mathcal{X}^*}(x)) \geq \alpha \|x - \Pi_{\mathcal{X}^*}(x)\|^2 \quad \forall\, x \in \operatorname{dom} \varphi.$$

Main Results

A Well-known Fact

Tâtonnement in a linear Fisher market is equivalent to subgradient descent on

$$\min_{p \in \mathbb{R}^m_{\geq 0}} \varphi(p) = \sum_{j=1}^m p_j - \sum_{i=1}^n B_i \log\left(\min_{k \in [m]} \frac{p_k}{v_{ik}}\right).$$

Price Lower and Upper Bounds

Lemma. Let $\tilde{p}$ be an m-dimensional vector where

$$\tilde{p}_j = \frac{B_{\min}}{4m} \max_i \frac{v_{ij}}{\|v_i\|_\infty}, \quad \text{for all } j \in [m].$$

Assume that we adjust prices in the linear Fisher market using Tâtonnement with $p^0 \geq \tilde{p}$ and $\eta < \frac{1}{2m} \min_j \tilde{p}_j$. Then, we have

- $p_j^t \geq \tilde{p}_j - 2m\eta$ for all $j \in [m]$ and $t \geq 0$;
- ℓ_2 norm of the excess demand $\|z(p^t)\| \leq \frac{B}{\min_j \tilde{p}_j - 2m\eta} + m := G$ for all $t \geq 0$;
- $p_j^t \leq \left(1 + \frac{\eta}{\tilde{p}_j - 2m\eta}\right) B$ for all $j \in [m]$ and $t \geq 0$.

Quadratic Growth

Lemma. The convex function $\varphi(p)$ satisfies the quadratic growth condition with modulus

$$\alpha = \min_j \frac{p_j^*}{2\left(1 + \frac{\eta}{\tilde{p}_j - 2m\eta}\right)^2 B^2},$$

where p^* denotes the unique equilibrium price vector.

⋆ Use an auxiliary function $h(p) = \sum_{j=1}^m p_j - \sum_{j=1}^m \sum_{i=1}^n p_j^* x_{ij}^* \log\left(\frac{p_j}{v_{ij}}\right)$.

Linear Convergence

Theorem. *Assume that we adjust prices in the linear Fisher market using Tâtonnement with $p^0 \geq \tilde{p}$ and $\eta < \frac{1}{2m} \min_j \tilde{p}_j$. Then, we have*

$$\|p^t - p^*\|^2 \leq (1 - 2\eta\alpha)^t \|p^0 - p^*\|^2 + e, \quad \textit{for all } t \geq 0,$$

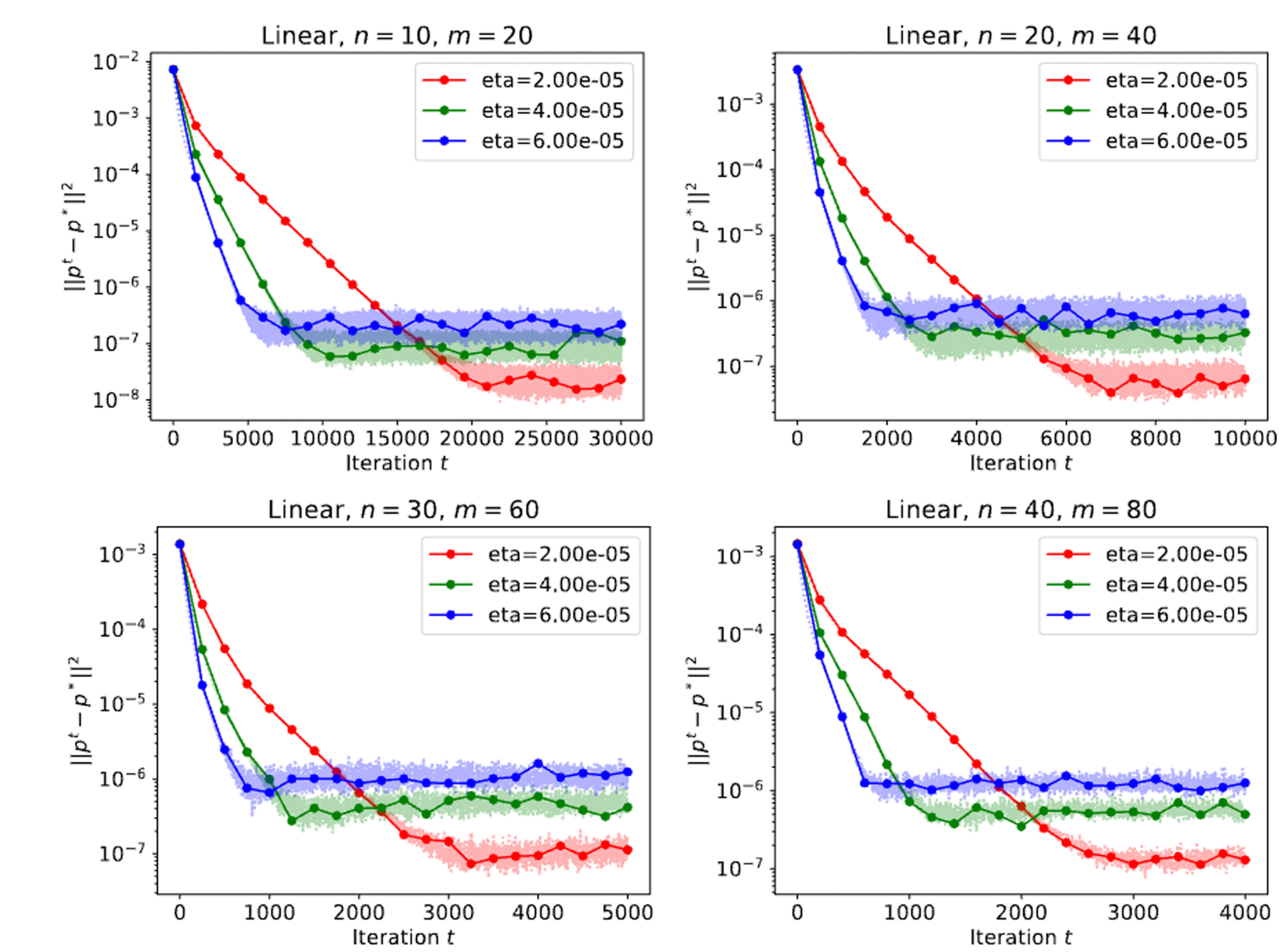
where

$$e = \frac{\eta G^2}{2\alpha} = \frac{\eta}{2\alpha}\left(\frac{B}{\min_j \tilde{p}_j - 2m\eta} + m\right)^2.$$

⋆ For any given $\epsilon > 0$, we have an η s.t. $\|p - p^*\|^2 \leq \epsilon$ in $\mathcal{O}(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$ steps.

Experiments

Convergence of squared error norms on random generated instances of different sizes



Zoom into the initial time steps of tâtonnement